

Degenerate Four Virtual Soliton Resonance for KP-II

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Abstract

By using dissipative version of the second and the third members of AKNS hierarchy, a new method to solve 2+1 dimensional Kadomtsev-Petviashvili (KP-II) equation is proposed. We show that dissipative solitons (dissipatons) of those members give rise to the real solitons of KP-II. From the Hirota bilinear form of the $SL(2, \mathbb{R})$ AKNS flows, we formulate a new bilinear representation for KP-II, by which, one and two soliton solutions are constructed and the resonance character of their mutual interactions is studied. By our bilinear form, we first time created four virtual soliton resonance solution for KP-II and established relations of it with degenerate four-soliton solution in the Hirota-Satsuma bilinear form for KP-II.

1 Introduction

Recently, dissipative version of AKNS hierarchy has been considered in connection with 1+1 dimensional (lineal) gravity models [1]. It is found that the second flow described by the dissipative version of the Nonlinear Schrödinger equation - the so called reaction-diffusion system, admits new soliton type solutions called *dissipatons*. Dissipatons have exponentially growing-decaying amplitudes, with perfect soliton shape for the bilinear product of them and the resonance interaction behaviour.

In the present paper we study resonance dissipative solitons in AKNS hierarchy and show that they give rise to the real solitons of KP-II. Our

approach is based on a new method to generate solutions of 2+1 dimensional KP equation: we show that if one considers a simultaneous solution of the second and the third flows from the AKNS hierarchy, then the product e^+e^- satisfies the KPII equation (Proposition 1). Using these results we construct new bilinear representation of KPII equation with one and two soliton solutions. We show that our two-soliton solution corresponds to the degenerate four soliton solution in the standard Hirota form of KP, and displays the four virtual soliton resonance.

2 SL(2,R) AKNS Hierarchy

The dissipative SL(2,R) AKNS hierarchy of evolution equations

$$\frac{1}{2}\sigma_3 \begin{pmatrix} e^+ \\ e^- \end{pmatrix}_{t_N} = \mathfrak{R}^{N+1} \begin{pmatrix} e^+ \\ e^- \end{pmatrix}, \quad (1)$$

where $N = 0, 1, 2, \dots$, ($\Lambda < 0$), is generated by the recursion operator \mathfrak{R}

$$\mathfrak{R} = \begin{pmatrix} \partial_x - \frac{\Lambda}{4}e^+ \int^x e^- & -\frac{\Lambda}{4}e^+ \int^x e^+ \\ -\frac{\Lambda}{4}e^- \int^x e^- & \partial_x + \frac{\Lambda}{4}e^- \int^x e^+ \end{pmatrix}. \quad (2)$$

Then, the second and third members of AKNS hierarchy appear as

$$\begin{cases} e_{t_1}^+ = e_{xx}^+ + \frac{\Lambda}{4}e^+e^-e^+ \\ -e_{t_1}^- = e_{xx}^- + \frac{\Lambda}{4}e^+e^-e^- \end{cases} \quad (3)$$

and

$$\begin{cases} e_{t_2}^+ = e_{xxx}^+ + \frac{3\Lambda}{4}e^+e^-e_x^+ \\ e_{t_2}^- = e_{xxx}^- + \frac{3\Lambda}{4}e^+e^-e_x^- \end{cases} \quad (4)$$

respectively. The first system (3), the dissipative version of the Nonlinear Schrödinger equation, is called the Reaction-Diffusion (RD) system [1]. It is connected with gauge theoretical formulation of 1+1 dimensional gravity, the constant curvature surfaces in pseudo-Euclidean space [1] and the NLS soliton problem in the quantum potential [1] [4].

3 Resonance Dissipatons in AKNS Hierarchy

3.1 Dissipatons of Reaction-Diffusion System

The second member of the AKNS (3), the Reaction-Diffusion equation, by substitution

$$e^\pm = \sqrt{\frac{8}{-\Lambda}} \frac{G^\pm(x, t)}{F(x, t)}, \quad (5)$$

admits the Hirota bilinear representation, $(t \equiv t_1)$,

$$(\pm D_t - D_x^2)(G^\pm \cdot F) = 0, \quad D_x^2(F \cdot F) = -2G^+G^-. \quad (6)$$

Then, any solution of this system determines a solution of the Reaction-Diffusion system (3). Simplest solution of bilinear system (6), has the form [4]

$$G^\pm = \pm e^{\eta_1^\pm}, \quad F = 1 + \frac{e^{(\eta_1^+ + \eta_1^-)}}{(k_1^+ + k_1^-)^2}, \quad (7)$$

where $\eta_1^\pm = k_1^\pm x \pm (k_1^\pm)^2 t + \eta_1^{\pm(0)}$. This solution determines soliton-type solution of the Reaction-Diffusion system with exponentially growing and decaying amplitudes, called the dissipaton [1]. But for the product e^+e^- one have the perfect one-soliton shape

$$e^+e^- = \frac{8k^2}{\Lambda \cosh^2[k(x - vt - x_0)]}, \quad (8)$$

of the amplitude $k = (k_1^+ + k_1^-)/2$, propagating with velocity $v = -(k_1^+ - k_1^-)$, where the initial position $x_0 = -\ln(k_1^+ + k_1^-)^2 + \eta_1^{+(0)} + \eta_1^{-(0)}$.

The Reaction-Diffusion system has a geometrical interpretation in a language of constant curvature surfaces [1]. It follows that when e^\pm satisfy the Reaction-Diffusion equations (3), the Riemannian metric describes two-dimensional pseudo-Riemannian space-time with constant curvature Λ : $R = g^{\mu\nu} R_{\mu\nu} = \Lambda$. If we calculate the metric for one dissipaton solution (7) it shows a singularity (sign changing) at $\tanh k(x - vt) = \pm v/2k$. This singularity (called the causal singularity) has physical interpretation in terms of black hole physics and relates with resonance properties of solitons. In fact, constructing two dissipaton solution we find that it describes a collision of two dissipatons creating the resonance (metastable) bound state [4].

3.2 Dissipative Solitons for the Third Flow

For the third flow of AKNS hierarchy we have the cubic dispersion system (4). The bilinear representation of this system for functions $e^\pm(x, t)$ in terms of three real functions G^\pm, F , as in (5), is

$$(D_t + D_x^3)(G^\pm \cdot F) = 0, \quad D_x^2(F \cdot F) = -2G^+G^-. \quad (9)$$

From the last equation we have expression for the product

$$U = e^+e^- = \frac{8}{-\Lambda} \frac{G^+G^-}{F^2} = \frac{4}{\Lambda} \frac{D_x^2(F \cdot F)}{F^2} = \frac{8}{\Lambda} \frac{\partial^2}{\partial x^2} \ln F. \quad (10)$$

Simplest solution of this system

$$G^\pm = \pm e^{\eta_1^\pm}, F = 1 + \frac{e^{(\eta_1^+ + \eta_1^-)}}{(k_1^+ + k_1^-)^2}, \quad (11)$$

where $\eta_1^\pm = k_1^\pm x - (k_1^\pm)^3 t + \eta_1^{\pm(0)}$, defines one dissipative soliton solution of the system (4)

$$e^\pm = \pm \sqrt{\frac{8}{-\Lambda}} \frac{|k_{11}^\pm|}{2} \frac{e^{\pm \frac{1}{2}(\eta_1^+ - \eta_1^-)}}{\cosh \frac{k_1^+ + k_1^-}{2} [x - vt - x_0]}, \quad (12)$$

where $v = k_1^{+2} - k_1^+ k_1^- + k_1^{-2}$, $x_0 = (\eta_1^{+(0)} + \eta_1^{-(0)})/k_1^+ k_1^-$, $\phi_{11} = -2 \ln k_{11}^{+-}$.

The system (4) admits following symmetric reduction: $e^+ = e^- = u$, leading to the MKdV equation

$$u_{t_2} = u_{xxx} + \frac{3\Lambda}{4} u^2 u_x. \quad (13)$$

Under this reduction, $k_1^+ = k_1^- \equiv k$, and the dissipative soliton (12) becomes one-soliton solution of MKdV

$$e^+ = e^- = u(x, t) = \sqrt{\frac{8}{-\Lambda}} \frac{|k|}{\cosh k(x - k^2 t - x_0)}. \quad (14)$$

This way we can see that dissipative soliton is a more general object reducible to the real soliton of MKdV. In a similar way two dissipative soliton solution of system (4) under reduction $k_1^+ = k_1^-$, $k_2^+ = k_2^-$, is reducible to two soliton solution of MKdV. The natural question is to find an evolution equation for $e^+ e^-$ product of dissipatons. As we show below it is KP-II equation in 2+1 dimensions.

4 KP-II Resonance Solitons

4.1 KP-II and AKNS hierarchy

AKNS hierarchy allows us to develop also a new method to find solution for (2+1) Kadomtsev-Petviashvili (KP) equation. Depending on sign of dispersion, two types of the KP equations are known. The minus sign in the right side of the KP corresponds to the case of negative dispersion and called KP-II. To relate KP-II with AKNS hierarchy let us consider the pair of functions $e^+(x, y, t)$, $e^-(x, y, t)$ satisfying the second and the third members of the

dissipative AKNS hierarchy. Here we renamed time variables t_1 as y and t_2 as t . Differentiating according to t and y , Eqs. (3) and (4) correspondingly, we can see that they are compatible.

Proposition 4.1.1 *Let the functions $e^+(x, y, t)$ and $e^-(x, y, t)$ are solutions of equations (3) and (4) simultaneously. Then the function $U(x, y, t) \equiv e^+ e^-$ satisfies the Kadomtsev-Petviashvili (KP II) equation*

$$(4U_t + \frac{3\Lambda}{4}(U^2)_x + U_{xxx})_x = -3U_{yy}. \quad (15)$$

Proof: We take derivative of U according to y variable and use Eq.(3), so that $U_y = (e_x^+ e^- - e_x^- e^+)_x$,

$$U_{yy} = (e_{xxx}^+ e^- + e_{xxx}^- e^+ - (e_x^+ e_x^-)_x) + \frac{\Lambda}{2} U_x U. \quad (16)$$

In a similar way for U_t we have

$$U_t = -(e_{xxx}^+ e^- + \frac{3\Lambda}{4} U e^- e_x^- + e_{xxx}^- e^+ + \frac{3\Lambda}{4} U e_x^- e^+), \quad (17)$$

$$U_{xt} = -(e_{xxx}^+ e^- + e_{xxx}^- e^+ + \frac{3\Lambda}{4} U U_x)_x. \quad (18)$$

Combining above formulas together

$$4U_{xt} + 3U_{yy} = [-e_{xxx}^+ e^- - e_{xxx}^- e^+ - \frac{3\Lambda}{2} U U_x - 3(e_x^+ e_x^-)_x]_x, \quad (19)$$

and using $U_{xxx} = e_{xxx}^+ e^- + e_{xxx}^- e^+ + 3e_{xx}^+ e_x^- + 3e_x^+ e_{xx}^-$, we get KP II (15)¹.

4.2 Bilinear Representation of KP II by AKNS flows

Using bilinear representations for systems (3) and (4) and Proposition 4.1.1 we can find bilinear representation for KP II. For RD system (3) bilinear form is given by (6), while for the third flow (4) by Eqs. (9).

Now we consider G^\pm and F as functions of three variables $G^{(\pm)} = G^{(\pm)}(x, y, t)$, $F = F(x, y, t)$, and require for these functions to be a solution of both bilinear systems (6), (9) simultaneously. Since the second equation in both systems is the same, it is sufficient to consider the next bilinear system

$$\begin{cases} (\pm D_y - D_x^2)(G^\pm \cdot F) = 0 \\ (D_t + D_x^3)(G^\pm \cdot F) = 0 \\ D_x^2(F \cdot F) = -2G^+ G^- \end{cases} \quad (20)$$

¹As Konopelchenko recently mentioned to us the similar results are known also as the symmetry reductions of KP [7],[8],[9]

Then, according to Proposition 4.1.1, any solution of this system generates a solution of KP II. From the last equation we can derive U directly in terms of function F only

$$U = e^+ e^- = \frac{8}{-\Lambda} \frac{G^+ G^-}{F^2} = \frac{4}{\Lambda} \frac{D_x^2(F \cdot F)}{F^2} = \frac{8}{\Lambda} \frac{\partial^2}{\partial x^2} \ln F \quad (21)$$

Simplest solution of this system

$$G^\pm = \pm e^{\eta_1^\pm}, \quad F = 1 + \frac{e^{(\eta_1^+ + \eta_1^-)}}{(k_1^+ + k_1^-)^2}, \quad (22)$$

where $\eta_1^\pm = k_1^\pm x \pm (k_1^\pm)^2 y - (k_1^\pm)^3 t + \eta_1^{\pm(0)}$, defines one-soliton solution of KP II according to Eq.(21)

$$U = \frac{2(k_1^+ + k_1^-)^2}{\Lambda \cosh^2 \frac{1}{2}[(k_1^+ + k_1^-)x + (k_1^{+2} - k_1^{-2})y - (k_1^{+3} + k_1^{-3})t + \gamma]}, \quad (23)$$

where $\gamma = -\ln(k_1^+ + k_1^-)^2 + \eta_1^{+(0)} + \eta_1^{-(0)}$. This soliton is a planar wave wall traveling in an arbitrary direction and called the line soliton.

4.3 Two Soliton Solution

Continuing Hirota's perturbation we find two soliton solution in the form

$$G^\pm = \pm(e^{\eta_1^\pm} + e^{\eta_2^\pm} + \alpha_1^\pm e^{\eta_1^+ + \eta_1^- + \eta_2^\pm} + \alpha_2^\pm e^{\eta_2^+ + \eta_2^- + \eta_1^\pm}), \quad (24)$$

$$F = 1 + \frac{e^{\eta_1^+ + \eta_1^-}}{(k_{11}^{+-})^2} + \frac{e^{\eta_1^+ + \eta_2^-}}{(k_{12}^{+-})^2} + \frac{e^{\eta_2^+ + \eta_1^-}}{(k_{21}^{+-})^2} + \frac{e^{\eta_2^+ + \eta_2^-}}{(k_{22}^{+-})^2} + \beta e^{\eta_1^+ + \eta_1^- + \eta_2^+ + \eta_2^-}, \quad (25)$$

where $\eta_i^\pm = k_i^\pm x \pm (k_i^\pm)^2 y - (k_i^\pm)^3 t + \eta_i^{\pm(0)}$, $k_{ij}^{ab} = k_i^a + k_j^b$, $(i, j = 1, 2)$, $(a, b = +-)$,

$$\alpha_1^\pm = \frac{(k_1^\pm - k_2^\pm)^2}{(k_{11}^{+-} k_{21}^{\pm\mp})^2}, \quad \alpha_2^\pm = \frac{(k_1^\pm - k_2^\pm)^2}{(k_{22}^{+-} k_{12}^{\pm\mp})^2}, \quad \beta = \frac{(k_1^+ - k_2^+)^2 (k_1^- - k_2^-)^2}{(k_{11}^{+-} k_{12}^{+-} k_{21}^{+-} k_{22}^{+-})^2}.$$

It provides two-soliton solution of KP II according to Eq.(21).

4.4 Degenerate Four-Soliton Solution

For KP II another bilinear form in terms of function F only is known [2]

$$(D_x D_t + D_x^4 + D_y^2)(F \cdot F) = 0 \quad (26)$$

Thus, it is natural to compare soliton solutions of our bilinear equations (20) with the ones given by this equation. To solve equation (26) we consider $F = 1 + \varepsilon F_1 + \varepsilon^2 F_2 + \dots$. The solution $F_1 = e^{\eta_1}$, where $\eta_1 = k_1 x + \Omega_1 y + \omega_1 t + \eta_1^0$, and dispersion $k_1 \omega_1 + k_1^4 + \Omega_1^2 = 0$ with $F_n = 0$, ($n = 2, 3, \dots$), under identification $k_1 = k_1^+ + k_1^-$, $\Omega_1 = \sqrt{3}(k_1^{+2} - k_1^{-2})$, $\omega_1 = -4(k_1^{+3} + k_1^{-3})$, and rescaling $4t \rightarrow t$, $\sqrt{3}y \rightarrow y$, determines one soliton solution of KP II (15). We realize that it coincides with our one soliton solution (23). But two soliton solution of equation (26) is not correspond to our two-soliton solution (24), (25). Appearance of four different terms $e^{\eta_i^{\pm} + \eta_k^{\pm}}$ in equation (25), suggest that our two-soliton solution should corresponds to some degenerate case of four soliton solution of Eq(26).² To construct four soliton solution first we find following solutions of bilinear equations (26)

$$F_1 = e^{\eta_1}, \quad F_2 = e^{\eta_2}, \quad F_4 = e^{\eta_3}, \quad (27)$$

where $\eta_i = k_i x + \Omega_i y + \omega_i t + \eta_i^0$, $i = 1, 2, 3$, dispersion

$$k_i \omega_i + k_i^4 + \Omega_i^2 = 0 \quad (28)$$

and

$$F_3 = \alpha_{12} e^{\eta_1 + \eta_2}, \quad F_5 = \alpha_{13} e^{\eta_1 + \eta_3}, \quad F_6 = \alpha_{23} e^{\eta_2 + \eta_3}, \quad (29)$$

where

$$\alpha_{ij} = -\frac{(k_i - k_j)(\omega_i - \omega_j) + (k_i - k_j)^4 + (\Omega_i - \Omega_j)^2}{(k_i + k_j)(\omega_i + \omega_j) + (k_i + k_j)^4 + (\Omega_i + \Omega_j)^2}, \quad i, j = 1, 2, 3. \quad (30)$$

Then we parameterize our solution in the form

$$\begin{aligned} k_1 &= k_1^+ + k_1^-, & \omega_1 &= -4(k_1^{+3} + k_1^{-3}), & \Omega_1 &= \sqrt{3}(k_1^{+2} - k_1^{-2}), \\ k_2 &= k_2^+ + k_2^-, & \omega_2 &= -4(k_2^{+3} + k_2^{-3}), & \Omega_2 &= \sqrt{3}(k_2^{+2} - k_2^{-2}), \\ k_3 &= k_1^+ + k_2^-, & \omega_3 &= -4(k_1^{+3} + k_2^{-3}), & \Omega_3 &= \sqrt{3}(k_1^{+2} - k_2^{-2}) \\ k_4 &= k_2^+ + k_1^-, & \omega_4 &= -4(k_2^{+3} + k_1^{-3}), & \Omega_4 &= \sqrt{3}(k_2^{+2} + k_1^{-2}), \end{aligned} \quad (31)$$

satisfying dispersion relations (28). Substituting these parameterizations to above solutions we find that

$$\alpha_{13} = 0 \Rightarrow F_5 = 0, \quad \alpha_{23} = 0 \Rightarrow F_6 = 0 \quad (32)$$

²One of the authors (O.P.) thanks Prof. J. Hietarinta for this suggestion

Continuing Hirota's perturbation with solution $F_7 = e^{\eta_4}$, where $\eta_4 = k_4x + \Omega_4y + \omega_4t + \eta_4^0$, we find that $F_8 = \alpha_{14}e^{\eta_1+\eta_4}$, where

$$\alpha_{14} = -\frac{(k_1 - k_4)(\omega_1 - \omega_4) + (k_1 - k_4)^4 + (\Omega_1 - \Omega_4)^2}{(k_1 + k_4)(\omega_1 + \omega_4) + (k_1 + k_4)^4 + (\Omega_1 + \Omega_4)^2} \quad (33)$$

and after the parameterizations given above (31) it also vanishes

$$\alpha_{14} = 0 \Rightarrow F_8 = 0 \quad (34)$$

The next solution $F_9 = \alpha_{24}e^{\eta_2+\eta_4}$, where

$$\alpha_{24} = -\frac{(k_2 - k_4)(\omega_2 - \omega_4) + (k_2 - k_4)^4 + (\Omega_2 - \Omega_4)^2}{(k_2 + k_4)(\omega_2 + \omega_4) + (k_2 + k_4)^4 + (\Omega_2 + \Omega_4)^2}, \quad (35)$$

also is zero

$$\alpha_{24} = 0 \Rightarrow F_9 = 0. \quad (36)$$

Then we have $F_{10} = 0$, and $F_{11} = \alpha_{34}e^{\eta_3+\eta_4}$, where

$$\alpha_{34} = -\frac{(k_3 - k_4)(\omega_3 - \omega_4) + (k_3 - k_4)^4 + (\Omega_3 - \Omega_4)^2}{(k_3 + k_4)(\omega_3 + \omega_4) + (k_3 + k_4)^4 + (\Omega_3 + \Omega_4)^2}. \quad (37)$$

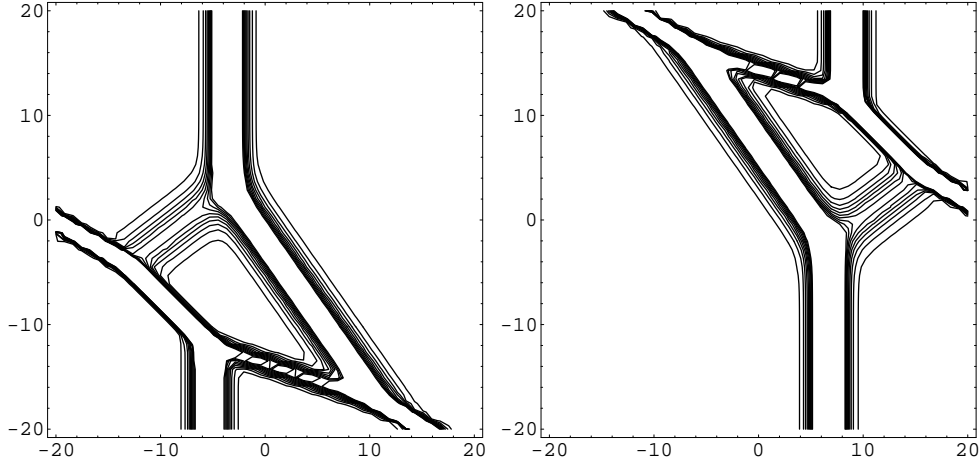
When it is checked for higher order terms we find that $F_{12} = F_{13} = \dots = 0$. Thus, we have degenerate four-soliton solution of equations (26) in the form

$$F = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + e^{\eta_4} + \alpha_{12}e^{\eta_1+\eta_2} + \alpha_{34}e^{\eta_3+\eta_4} \quad (38)$$

Comparing this solution with the one in Eq.(25) and taking into account that according parameterizations (31), $\eta_1 + \eta_2 = \eta_3 + \eta_4$, we see that they are coincide. The above consideration shows that our two-soliton solution of KP-II corresponds to the degenerate four soliton solution in the canonical Hirota form (26). Moreover, it allows us to find new four virtual soliton resonance for KP-II.

4.5 Resonance Interaction of Planar Solitons

Choosing different values of parameters for our two soliton solution we find resonance character of soliton's interaction. For the next choice of parameters $k_1^+ = 2, k_1^- = 1, k_2^+ = 1, k_2^- = 0.3$, and vanishing value of the position shift constants, we obtained two soliton solution moving in the plane with constant velocity, with creation of the four, so called virtual solitons (solitons without asymptotic states at infinity).



(a) Fig.1a

(b) Fig.1b

The resonance character of our planar soliton interactions is related with resonance nature of dissipatons considered in Section 3. It has been reported also in several systems, but the four virtual soliton resonance does not seem to have been done for KP-II [6] prior to our work. In Conference on Nonlinear Physics, Gallipoli, June-July 2004, we realized that resonance solitons for KP-II very recently have been constructed also by Biondini and Kodama [11], [12] using Sato's theory. Then, the comparison shows that our bilinear constraint plays the similar role as the Toda lattice in their paper.

5 Conclusions

The idea to use couple of equations from the AKNS hierarchy to generate a solution of KP, can be applied also to multidimensional systems with zero curvature structure as the Chern-Simons gauge theory. Our three dimensional zero curvature representation of KP-II gives flat non-Abelian connection for $SL(2, R)$ and corresponds to a sector of three dimensional gravity theory. Recently, we have shown that idea similar to the one presented in this paper can be applied also for Kaup-Newell hierarchy. In this case, combining the second and the third flows of dissipative version of derivative NLS (DNLS) we found resonance soliton dynamics for modified KP-II [10].

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